

$$f(x) = x^2$$

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{(x+h)^2 - x^2}{h} \\ &= \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \frac{2xh + h^2}{h} \\ &= \frac{2xh}{h} + \frac{h^2}{h} \\ &= 2x + h\end{aligned}$$

THUS,

$$\frac{f(x+h) - f(x)}{h} = 2x + h = \begin{array}{l} \text{"SLOPE OF SECANT"} \\ \text{FROM } x \\ \text{TO } x+h \end{array}$$

FOR EXAMPLE, IF WE WANTED THE SLOPE OF THE SECANT FROM 3 TO 10 THEN WE \rightarrow $x+h=10$ WOULD USE $x=3$ AND $h=7$ TO GET SLOPE $= 2(3) + 7 = 13$

①

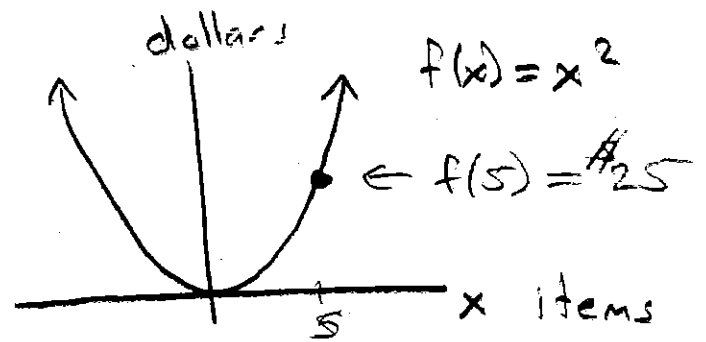
WE WANT TO KNOW THE SLOPE AS $h \rightarrow 0$ TO GET

$$f'(x) = 2x = \begin{array}{l} \text{"SLOPE OF} \\ \text{THE TANGENT} \\ \text{LINE AT } x \end{array}$$

NOTES/COMMENTS

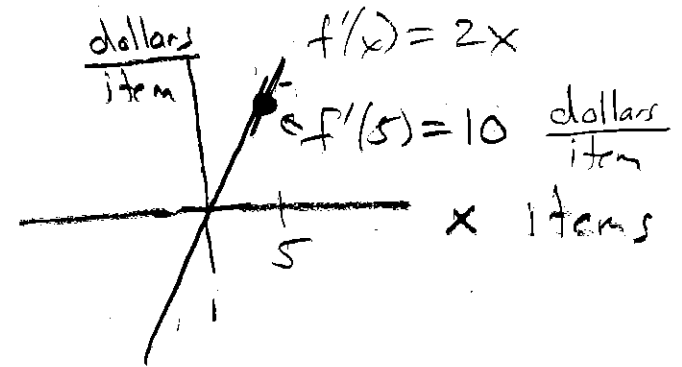
WE JUST FOUND

IF $y = f(x) = x^2$ \leftarrow HEIGHT ON GRAPH AT x
THEN $y' = f'(x) = 2x$ \leftarrow SLOPE OF TANGENT AT x



ASSUME $f(x) = x^2$ dollars in revenue
WHERE x is in items

THEN $f'(x) = 2x$ is $\frac{\text{dollars}}{\text{item}}$



FOR EXAMPLE, IF YOU SELL $x = 5$ ITEMS

THEN $f(5) = (5)^2 = 25$ dollars in revenue

AND $f'(5) = 2(5) = 10$ $\frac{\text{dollars}}{\text{item}}$

Let's try it again:

$$f(x) = x^3$$

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^3 - x^3}{h}$$

$$= \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3}}{h}$$

$$= \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= 3x^2 + 3xh + h^2$$

LETTING $h \rightarrow 0$

GIVES

$$f'(x) = 3x^2$$

SIMILARLY, $f(x) = x^4$

$$\frac{(x+h)^4 - x^4}{h} = \frac{\cancel{x^4} + 4x^3h + \text{"STUFF"} - \cancel{x^4}}{h}$$

$$= 4x^3 + \text{"STUFF"} \text{ AN } h \text{ IN IT}$$

SO $f'(x) = 4x^3$ AND SO ON...

$$(x+h)^3 = (x+h)(x+h)(x+h)$$

$$= (x^2 + 2xh + h^2)(x+h)$$

$$= x^3 + 2x^2h + xh^2 + x^2h + 2xh^2 + h^3$$

$$= x^3 + 3x^2h + 3xh^2 + h^3$$

ASIDE PASCAL'S TRIANGLE

$$(x+h)^2 = x^2 + 2xh + h^2$$

$$(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$$

$$(x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$$

etc...

← EACH TERM HAS AN h IN IT

POWER RULE: If $f(x) = x^n$, then $f'(x) = n x^{n-1}$.

Written briefly, $\frac{d}{dx}(x^n) = n x^{n-1}$.

Special Cases:

$$\frac{d}{dx}(x) = 1.$$

$$\frac{d}{dx}(1) = 0.$$

Note: Although we won't prove this.

The power rule works for ALL powers

(including negative and decimal powers)

Examples

$$\frac{d}{dx}(x^2) = 2x^1$$

$$\frac{d}{dx}(x^3) = 3x^2$$

$$\frac{d}{dx}(x^4) = 4x^3$$

$$\frac{d}{dx}(x^5) = 5x^4$$

$$\frac{d}{dx}(x^6) = 6x^5$$

etc.

$$\frac{d}{dx}(x^{-1}) = -x^{-2}$$

$$\frac{d}{dx}(x^{-2}) = -2x^{-3}$$

$$\frac{d}{dx}(x^{-3}) = -3x^{-4}$$

$$\frac{d}{dx}(x^{2.5}) = 2.5x^{1.5}$$

$$\frac{d}{dx}(x^1) = 1x^0 = 1$$

$$\frac{d}{dx}(1) = \frac{d}{dx}(x^0) = 0$$

Exponents Review

$$\frac{1}{x^b} = x^{-b}$$

EXAMPLE 1

$$f(x) = \frac{1}{x^2} = x^{-2} \Rightarrow f'(x) = -2x^{-3}$$

$$f(x) = \frac{1}{x^{1.2}} = x^{-1.2} \Rightarrow f'(x) = -1.2x^{-2.2}$$

$$f(x) = \frac{1}{x^{-3}} = x^3 \Rightarrow f'(x) = 3x^2$$

$$\sqrt[a]{x} = x^{1/a} \text{ and } \sqrt{x} = x^{1/2}$$

$$f(x) = \sqrt{x} = x^{1/2} \Rightarrow f'(x) = \frac{1}{2}x^{-1/2}$$

$$f(x) = \sqrt[3]{x} = x^{1/3} \Rightarrow f'(x) = \frac{1}{3}x^{-2/3}$$

$$f(x) = \sqrt[5]{x} = x^{1/5} \Rightarrow f'(x) = \frac{1}{5}x^{-4/5}$$

$$x^a x^b = x^{a+b}$$

$$\frac{x^a}{x^b} = x^{a-b}$$

$$(x^a)^b = x^{ab}$$

$$f(x) = \frac{x^4 x^2}{x} = \frac{x^6}{x} = x^5 \Rightarrow f'(x) = 5x^4$$

$$f(x) = \sqrt{x^3} = (x^3)^{1/2} = x^{3/2} \Rightarrow f'(x) = \frac{3}{2}x^{1/2}$$

$$f(x) = \sqrt[3]{x^7} = x^{7/3} \Rightarrow f'(x) = \frac{7}{3}x^{4/3}$$

SUM/DIFFERENCE RULE: $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$, and
 $\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$.

Examples

$$\frac{d}{dx}(x^{10} + x^9) = 10x^9 + 9x^8$$

$$\frac{d}{dx}\left(\sqrt[3]{x} + \frac{1}{x^2}\right) = \frac{d}{dx}\left(x^{1/3} + x^{-2}\right) = \frac{1}{3}x^{-2/3} - 2x^{-3}$$

$$\frac{d}{dx}(x^{11} + 7 + x) = 11x^{10} + 0 + 1 = 11x^{10} + 1$$

$$\text{COEFFICIENT RULE: } \frac{d}{dx}(cf(x)) = cf'(x).$$

Special Cases:

$$\frac{d}{dx}(cx) = c.$$

$$\frac{d}{dx}(c) = 0.$$

EXAMPLES

$$\frac{d}{dx}(4x^3) = 4 \cdot 3x^2 = 12x^2$$

$$\frac{d}{dx}(-14x^{100}) = -14 \cdot 100x^{99} = -1400x^{99}$$

$$\frac{d}{dx}(3x^{10} - 15x^3) = 3 \cdot 10x^9 - 15 \cdot 3x^2 = 30x^9 - 45x^2$$

$$\frac{d}{dx}(6x + 4) = 6 + 0 = 6$$

Derivative methods so far:

1. **Expand** into sum of terms.
2. **Rewrite** each term as: cx^b .
3. Bring coefficients/sum along for the ride.
4. Use power rule.
5. Simplify.

FIND THE DERIVATIVE

$$1. y = 5x - 3x^2 + 1$$

$$\frac{dy}{dx} = \underset{\downarrow}{5} - 3 \cdot \underset{\downarrow}{2}x + \underset{\downarrow}{0} = \boxed{5 - 6x}$$

$$2. R(q) = -0.4q^3 + \frac{q^2}{2} + 4.5q = -0.4q^3 + \frac{1}{2}q^2 + 4.5q$$

$$R'(q) = -0.4 \cdot 3q^2 + \frac{1}{2} \cdot 2q + 4.5 = \boxed{-1.2q^2 + q + 4.5}$$

$$3. y = \sqrt[3]{x} - 3x^4 + \frac{5}{\sqrt{x}} = x^{1/3} - 3x^4 + 5x^{-1/2}$$

$$\frac{dy}{dx} = \frac{1}{3}x^{-2/3} - 12x^3 - \frac{5}{2}x^{-3/2}$$

$$4. f(x) = x^3 \left(x^5 + \frac{2}{x} \right) = x^3 \cdot x^5 + x^3 \cdot \frac{2}{x} = x^8 + 2x^2$$

$$f'(x) = 8x^7 + 4x$$

$$5. g(x) = 12\sqrt{x} - \frac{10}{x^2} + 17 = 12x^{1/2} - 10x^{-2} + 17$$

$$g'(x) = 6x^{-1/2} + 20x^{-3} + 0$$

$$6. y = \frac{x^{-2} + x^7 - 2}{\sqrt{x}} = \frac{x^{-2}}{x^{1/2}} + \frac{x^7}{x^{1/2}} - \frac{2}{x^{1/2}} = x^{-2.5} + x^{6.5} - 2x^{-1/2}$$

$$\frac{dy}{dx} = -2.5x^{-3.5} + 6.5x^{5.5} - x^{-3/2}$$

$$7. y = \sqrt{x}(x^3 + 4) = x^{1/2}x^3 + 4x^{1/2} = x^{3.5} + 4x^{1/2}$$

$$\frac{dy}{dx} = 3.5x^{2.5} + 2x^{-1/2}$$

$$8. y = \frac{x^3}{3} + \frac{5}{x^2} + 6\sqrt[3]{x^2} = \frac{1}{3}x^3 + 5x^{-2} + 6x^{2/3}$$

$$\frac{dy}{dx} = x^2 - 10x^{-3} + 6 \cdot \frac{2}{3} x^{-1/3} = \boxed{x^2 - 10x^{-3} + 4x^{-1/3}}$$

$$9. y = \frac{t^2 - \sqrt{t} + 2}{t^2} = \frac{t^2}{t^2} - \frac{t^{1/2}}{t^2} + \frac{2}{t^2} = 1 - t^{-1.5} + 2t^{-2}$$

$$\frac{dy}{dx} = 0 + 1.5 t^{-2.5} - 4t^{-3}$$

$$10. y = \frac{4\sqrt[3]{x^2}}{5\sqrt{x^3}} = \frac{4}{5} \frac{x^{2/3}}{x^{3/2}} = \frac{4}{5} x^{(\frac{2}{3} - \frac{3}{2})} \quad \frac{2}{3} - \frac{3}{2} = \frac{4}{6} - \frac{9}{6} = -\frac{5}{6}$$

$$= \frac{4}{5} x^{-5/6}$$

$$\frac{dy}{dx} = \frac{4}{5} \left(-\frac{5}{6}\right) x^{-11/6} = -\frac{2}{3} x^{-11/6}$$